On Trigonometrical Differences of Altitude.
By G. Tyrrell McCaw, M.A.

In the present methods in use in practical geodesy for determining the difference of altitude of two stations on the Earth's surface, it is common to assume that the vertical refraction at both ends of the line joining two stations is the same. Sometimes even the error seems to be made of implicitly assuming that refraction is eliminated by simultaneous and reciprocal observations at the two ends.

The refraction at each station is commonly regarded as a linear factor of the "contained angle" subtended by the stations at the Earth's centre; or, in other words, is considered as being directly proportional to the distance between the two stations projected on the geodetic surface. The refraction r is therefore expressed thus—

$$r = k \, . \, \psi,$$

ψ being the contained angle and k the so-called "coefficient of refraction."*

Numerous experiments in various parts of the world have shown that this coefficient is by no means constant, not even for the same station. Thus in the United Kingdom, though in 72 cases the value lay between 0·0733 and 0·0804, among 144 cases the values varied between the extremes of 0·0320 and 0·1058. Even in the clear air of mountain heights, where one usually obtains consistent results, there may be wide differences. Thus in the line Ben Lomond to Ben Nevis, the value on Ben Lomond was 0·0719, while on Ben Nevis it was 0·0942.

In Western America, when the refraction at a single station was investigated, the coefficient was found to have a diurnal range as great as 0·04 in some cases. Periods of rapid change occurred about an hour before sunrise and an hour after sunset. The time of *maximum* was about 3 a.m., and the time of *minimum* varied at different stations between 10 a.m. and 4 p.m., the hour from 2 to 3 p.m. marking apparently the mean period. On the line Mt. Diablo—Martinez East, the value at 3 p.m. at the former station (height 1173 metres) was 0·0653, and at the latter (57 m.) 0·1007, while at the latter the value rose to 0·1415 at 3 a.m. The value for the interior of the continent was about 0·065, over parts of the sea near the Atlantic coast about 0·078, and over parts of the sea near the Pacific coast about 0·085. Generally speaking, low elevations gave higher values and greater range; higher values were also obtained in the excessive moisture of the sea-coast. The range of temperature seems chiefly to determine the form of the curve which represents the diurnal variation of the coefficient; height or moisture, its position.

* In Continental practice it is the custom to apply the symbol k to the ratio of the total refraction to the contained angle.

Taking the mean value of the two stations on a line, the Coast Survey obtained the following results:—

	Jackson Butte (714 m.)— Round Top (3174 m.) (Inland).	Sea Horizon— Ragged Mt. (397 m.) (Atlantic coast).	Martinez East (57 m.)— Mt. Diablo (1173 m.) (Pacific ocean).
Daily Mean,	0·0646	0·0849	0·1000
Diurnal Range,	0·0168	0·0293	0·0333

In the computations of the great American Arc of Parallel 39°, the mean coefficients for the various lines in the western portion of the arc were tabulated, the mean heights of the stations and mean temperatures also being recorded. These results seemed to suggest an empirical formula of the form—

$$k = k_0 + (t - t_0)x + (h - h_0)y,$$

t_0 and h_0 being the mean temperature (Centigrade) and height (in hectometres) of all the stations in a given area of the triangulation.

For the more eastern area $k_0 = 0·0534$, $t_0 = 9°·9$, $h_0 = 37·7$
" " western " $= 0·0578$, $= 11°·2$, $= 32·5$

Forming the equations for all the observations in each area, the following values of x and y were obtained by mean squares:—

For the more eastern area $x = 0·00036$, $y = 0·00087$
" " western " $= 0·00081$, $= 0·00091$

The agreement between the observed and computed values was, in general, fairly good.

Over the hot plains of India much greater discrepancies were observed. In one case a depression of $4' 52''·6$ at 4.30 p.m. changed to an elevation of $2' 24''·0$ at 10.50 p.m., the stations being only 10·5 miles apart. The refraction had sometimes a negative value, and signals which were invisible during the day were seen above the horizon at sunset. The coefficient in India actually ranged from $-0·09$ to $+1·21$.

In Cape Colony and Natal the refraction was more constant, the extreme values of the coefficient being found to be 0·045 and 0·106, but by no means all the lines were computed. The value was seen to increase by about 0·00045 per mile for distances between 20 and 60 miles—a result which shows that the refraction is not a linear factor of the distance.

Careful observations in France indicated that refraction there is greatest about daybreak, and then diminishes very rapidly till 8 a.m., and from that very slowly till 10 a.m. Thence the refraction remains nearly constant till 4 p.m., after which it begins again to increase.

Very few experiments have been made with a view to determine seasonal variation, but there is evidence enough to show that refraction changes very considerably with the season.

Enough has been said to prove how very variable the coefficient may be. It is otherwise obvious, *à priori*, that the refraction at a

point in a hot, low, moist plain must be very different from that on a cool, dry, mountain height. Indeed there is a very special reason for this refraction remaining even so constant as it is : in primary triangulation work, practically all measurement of vertical angle is made within the range of 87° and 93° zenith distance, and the great majority of the observations are confined within 89° 45′ and 90° 45′—a range of only one degree.

One of the practical methods devised to overcome the difficulty of the variability of refraction is that which assumes that the coefficient is constant for a given station rather than for the extremities of a line. This is also a fairly extravagant hypothesis. It ignores the variation of zenith distance with which the refraction is doubtless in some way connected; it ignores the varying length of the lines by which the refraction is affected in some measure, as shown in South Africa; it ignores, further, those fortuitous variations which take place from moment to moment, due to rain, clouds, wind, heat waves, etc., the effect of which in an individual measure may have a preponderating effect on the mean of a series of observations.

The actual method of determining the coefficients on this assumption is as follows:—If $k_1, k_2, k_3, \ldots\ldots$ be the mean values of the coefficients at the stations $P_1, P_2, P_3 \ldots\ldots$ respectively, then it is easily seen that these coefficients are connected by equations of the form

$$k_1 + k_2 = \text{a constant},$$

the constant being determined from the observations; forming the equations for all the stations in a circuit, the values of $k_1, k_2, k_3 \ldots\ldots$ are determined.

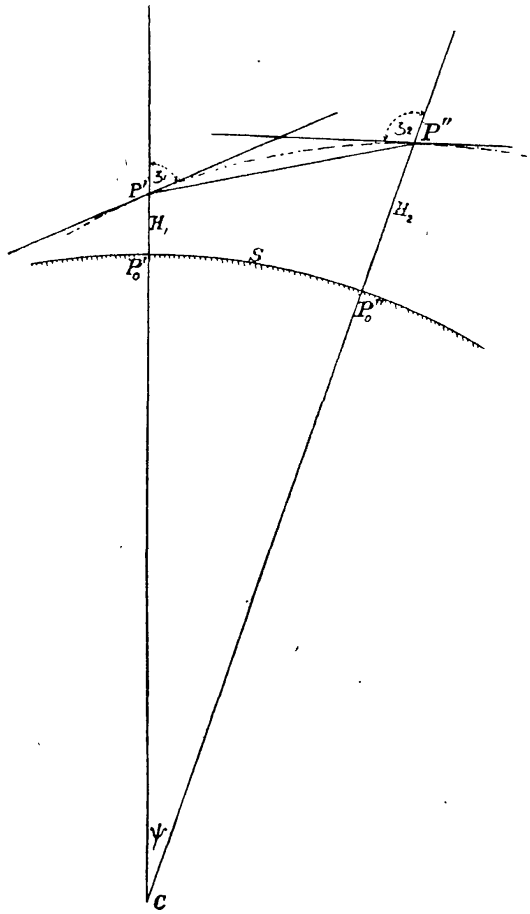
It is, of course, well recognised that it is an entirely gratuitous assumption to express the refraction at either end of a line, or the total refraction, as simply proportional to the arc between the two stations. It is, therefore, in more refined methods, usual to endeavour to find an expression for k which will introduce the necessary meteorological conditions. These investigations lead to equations almost unmanageable in practice, and depend on atmospheric hypotheses not generally recognised.

In all present methods, so far as the writer is aware, there are assumptions which can with difficulty be defended on theoretical grounds, or on the ground that the coefficients adopted are found by experience to give concordant results. Recognising this fact, the Seventh Congress of the International Geodetic Association, in 1883, after hearing a report from Dr. Bauernfeind, expressed the desire that the various States which contribute to the Association should undertake exact investigations to determine the influence of the varying conditions of ground and climate on the terrestrial refraction.

As the problem remains unsolved, the writer offers the following theoretical solution, based on generally accepted physical

laws and recognised hypotheses. In this solution it will be seen that the use of the coefficient k has been entirely abandoned.

It is necessary to premise at the outset that the observations at the ends of the line are reciprocal and simultaneous. The temperature and pressure must be observed at both stations with standardised instruments. For the present no consideration will be taken of the effect of aqueous vapour; the influence of this vapour may be subsequently investigated.



In the figure, P', P'' are two stations on the Earth's surface; P_0', P_0'' their projections on the ellipsoid of reference; the mean radius of curvature of the ellipsoid along the line between P_0' and P_0'' is R; Z_1, Z_2 are the true, and ζ_1, ζ_2 the observed, zenith distances of the mutually observed points; the contained angle $P'OP''$ is ψ; the length of the arc $P_0'\ P_0''$ is S; H_1, H_2 are the heights of the two stations above the spheroid, and $H_2 - H_1 = h$, the height-difference.

We adopt the recognised hypothesis that the air in its normal condition consists of spherical strata, the density of each layer being constant, and varying continuously from layer to layer. Then, if a beam of light pass through media of continuously

varying density, the density and refractive index of a layer being any function whatever of the distance from a fixed point, we have the well-known relation

$$\mu . r . \sin \phi = a \ constant,$$

where

μ is the absolute index of refraction of any stratum,
r the distance of that stratum from the fixed origin,
ϕ the angle between the vector r and the tangent to the path of the beam at the point where it cuts the stratum whose index is μ.

Applying this law to the present case

$$\mu_1(R+H_1) \sin \zeta_1 = \mu_2(R+H_2) \sin \zeta_2$$

whence

$$\frac{R+H_2}{R+H_1} = \frac{\mu_1 \sin \zeta_1}{\mu_2 \sin \zeta_2} \quad . \quad . \quad . \quad . \quad (1)$$

Put the latter $= \omega$.

Then $$H_2 - H_1 = h = (R+H_1)(\omega - 1) \quad . \quad . \quad (2)$$

Thus the height-difference is known if we know ω, that is, if the ratio $\mu_1 : \mu_2$ is known.

Before proceeding to investigate this ratio we shall consider two particular cases.

From the geometrical conditions of the figure we have obviously

$$\frac{R+H_2}{R+H_1} = \frac{\sin Z_1}{\sin Z_2},$$

whence

$$H_2 - H_1 = h = 2(R+H_1) \frac{\sin \frac{1}{2}(Z_1 - Z_2) \cos \frac{1}{2}(Z_1 + Z_2)}{\sin Z_2} \quad (3)$$

which reduces to

$$h = 2(R+H_1) \sin \tfrac{1}{2}\psi . \frac{\sin \frac{1}{2}(Z_2 - Z_1)}{\cos \frac{1}{2}(Z_2 - Z_1 + \psi)} \quad . \quad (4)$$

which is the rigorous expression for the height-difference if the difference of the true zenith distances be known.

Consider first the particular case where the refraction at both ends of the line is the same.

Then, since $Z_1 - \zeta_1 = Z_2 - \zeta_2$, equation (4) becomes

$$h = 2(R+H_1) \sin \tfrac{1}{2}\psi . \frac{\sin \frac{1}{2}(\zeta_2 - \zeta_1)}{\cos \frac{1}{2}(\zeta_2 - \zeta_1 + \psi)} \quad . \quad (5)$$

which is the rigorous expression for h when the above assumption is supposed to hold good, the plumbline being uninfluenced by local attraction at either station, and the mean surface of curvature of the geoid between the two stations being regarded as part of a sphere.

Again, suppose that $\mu_1 = \mu_2$, *i.e.* that the index of refraction at both stations is the same; then equation (1) becomes

$$\frac{R+H_2}{R+H_1} = \frac{\sin \zeta_1}{\sin \zeta_2},$$

wherefore the height-difference is expressed by the equation (3), in which the observed zenith distance ζ is substituted for the true zenith distance Z.

To proceed with the general investigation, it is necessary to determine the ratio $\mu_1 : \mu_2$.

Suppose that a ray of light passes from a vacuum into a gas whose absolute index of refraction is μ and density ρ. Then if the density be varied, the index of refraction varies according to the formula

$$\mu - 1 = c.\rho, \text{ where } c \text{ is a constant} \quad . \quad . \quad (6)$$

This, the well-known law of Gladstone and Dale, is that which is generally accepted for solids, liquids, and gases. In the case of gases, however, where μ is nearly equal to unity, some have supposed that

$$\mu^2 - 1 = c.\rho$$

more nearly represents the true law.* We shall adopt Gladstone's law; but even should a somewhat different formula be discovered in the future, it is obvious that the present investigation is readily modifiable accordingly.

It only remains to find the density in terms of the temperature and pressure according to the usual relations dependent on the Boyle-Marriotte-Charles law.

Let p_1, p_2 be the atmospheric pressures at the stations P', P''.
t_1, t_2 ,, ,, temperatures ,, ,,
b_1, b_2 ,, heights of the barometers ,, ,,
g_1, g_2 ,, accelerations of gravity ,, ,,
τ_1, τ_2 ,, temperatures indicated by the thermometers attached to the mercurial columns at P', P'' when the corresponding atmospheric temperatures are t_1, t_2.
σ_0 be the density of mercury at 0° C.
β ,, coefficient of expansion of mercury.

Then we have the following relations:

$$\begin{aligned} p_1 &= \kappa.\rho_1.(1 + \alpha.t_1) \\ p_2 &= \kappa.\rho_2.(1 + \alpha.t_2) \end{aligned} \quad . \quad . \quad . \quad (7)$$

But

$$\begin{aligned} p_1 &= g_1.\sigma_0.(1 - \beta.\tau_1).b_1 \\ p_2 &= g_2.\sigma_0.(1 - \beta\,\tau_2).b_2 \end{aligned} \quad . \quad . \quad . \quad (8)$$

Accordingly

$$\begin{aligned} \rho_1 &= \frac{g_1.\sigma_0.(1 - \beta.\tau_1).b_1}{\kappa(1 + \alpha.t_1)} \\ \rho_2 &= \frac{g_2.\sigma_0.(1 - \beta.\tau_2).b_2}{\kappa(1 + \alpha.t_2)} \end{aligned} \quad . \quad . \quad . \quad (9)$$

* This latter relation is attributed to Biot and Arago.

To put these into more workable forms, let T_1, T_2 be the absolute temperatures, so that

$$T_1 = \frac{1}{\alpha} + t_1, \quad T_2 = \frac{1}{\alpha} + t_2;$$

and let

$$\mathfrak{T}_1 = \frac{1}{\beta} - \tau_1, \quad \mathfrak{T}_2 = \frac{1}{\beta} - \tau_2;$$

then

$$\omega = \frac{\mu_1 \sin\zeta_1}{\mu_2 \sin\zeta_2} = \frac{(1 + c.\rho_1).\sin\zeta_1}{(1 + c.\rho_2).\sin\zeta_2}.$$

Substituting ρ_1 and ρ_2 from (9).

$$\omega = \frac{\kappa\alpha + c\sigma_0\beta g_1 b_1 \dfrac{\mathfrak{T}_1}{T_1}}{\kappa\alpha + c\sigma_0\beta g_2 b_2 \dfrac{\mathfrak{T}_2}{T_2}} \cdot \frac{\sin\zeta_1}{\sin\zeta_2} \quad . \quad . \quad (10)$$

But g_1, g_2 may generally be put equal to g, where g is computed in the first instance by the formula

$$g = g_0 . \frac{R^2}{(R+H)^2},$$

g_0 being the acceleration of gravity at sea-level at the mean latitude of the survey, and H the average height of the stations. This will usually be sufficiently exact if for no other reason than that g_1, g_2 will often be considerably affected by the distribution of Earth-mass in the vicinity of the stations. Hence putting

$$\frac{\kappa.\alpha}{c.\sigma_0.\beta.g} = A,$$

we have finally

$$\omega = \frac{A + b_1 . \dfrac{\mathfrak{T}_1}{T_1}}{A + b_2 . \dfrac{\mathfrak{T}_2}{T_2}} \cdot \frac{\sin\zeta_1}{\sin\zeta_2} \quad . \quad . \quad . \quad (11)$$

If the units throughout be centimetres and degrees Centigrade,

A is about 5,262,250 cms.

Thus ω is completely determined from the observed zenith distances and the temperatures and pressures at the two stations. This value of ω substituted in equation (2)

$$h = (R + H_1)(\omega - 1)$$

gives the difference of height of the two stations, subject to the usual correction for "eye and object."

. . . .

It is obvious that, if the air is not in its normal condition, if it is locally disturbed by storms, rain, cloud, excessive moisture, and such-like, the distribution of density in spherical strata may be considerably upset, and the theory will no longer hold good. Should, however, such abnormal conditions occur, they will generally reveal themselves at one or both of the stations by considerable perturbations of the thermometer or barometer. Moreover, it must be remembered that the theory of celestial refraction is vitiated by the same disturbances, though it must be granted that these disturbances are more likely to occur in the strata near the surface, and in which geodetic observations take place, than in the upper regions of the atmosphere. Where long lines pass very close to the Earth's surface, or to intervening plateaus or mountain ranges, considerable discordances may be expected. It must be remembered, however, that conditions such as these, which frequently occur in practical geodesy, must upset any theory. It cannot, therefore, be expected that any theory of terrestrial refraction will yield results so accurate as the Besselian theory of celestial refraction. Still, much better results must follow from expressions based on acknowledged principles rather than blind hypotheses.

For a full account of the previous treatment of the subject, the reader may consult the following references. For drawing his attention to the last publication on this list, the writer would desire to acknowledge his indebtedness to Major E. H. Hills, C.M.G.

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Comparisons of the places of Mars calculated from Newcomb's Tables, with the places calculated from Le Verrier's Tables, near the times of Opposition in 1907 *and* 1909. By A. M. W. Downing, D.Sc., F.R.S.

These comparisons have been made by reducing the quantities given in the *Nautical Almanac* from Greenwich noon to Paris noon at convenient intervals, and then finding the differences between the reduced quantities and the corresponding quantities in the *Connaissance des Temps*.

It will be noticed that at these oppositions, when the planet is nearly at its least possible distance from the Earth, the differences in the heliocentric places derived from the two sets of Tables are much magnified when the transformation to geocentric places is effected. Also it must be remembered that different Tables of the Sun (Newcomb's and Le Verrier's respectively) are employed in making the transformation in the two cases. It will be remarked, however, that the difference in heliocentric longitude, which vanishes in 1907 September, is as much as 4″ in 1909 September. It appears desirable, at the present time, to draw attention to this discordance, as well as to the corresponding discordances in the geocentric places, which amount to comparatively large quantities at the epochs considered.

Mars 1907—*Corrections to Le Verrier's Tables.*

Day 1907.		R.A. Time.	R.A. Arc.	Decl.
		s	″	″
May	26	+·16	+2·2	−0·8
June	3	·17	2·3	0·8
	11	·23	3·1	0·6
	19	·28	3·8	0·8
	27	·28	3·8	1·2
July	5	·29	3·9	1·4
	13	·28	3·7	1·2
	21	·25	3·4	1·8
	29	·19	2·5	1·7
Aug.	6	·14	1·9	1·6
	14	+·13	+1·7	−1·1

On July 5 (near the time of Opposition):—

the correction to Le Verrier's heliocentric longitude of Mars is + 0″·7;
the correction to Le Verrier's longitude of the Sun is - 0″·5;
the distance of Mars from the Earth is 0·41.